

Finite good filtration dimension for modules over an algebra with good filtration.

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Dedicated to Eric Friedlander on his 60th birthday

Abstract

Let G be a connected reductive linear algebraic group over a field k of characteristic $p > 0$. Let p be large enough with respect to the root system. We show that if a finitely generated commutative k -algebra A with G -action has good filtration, then any noetherian A -module with compatible G -action has finite good filtration dimension.

1 Introduction

Consider a connected reductive linear algebraic group G defined over a field k of positive characteristic p . We say that G has the cohomological finite generation property (CFG) if the following holds: Let A be a finitely generated commutative k -algebra on which G acts rationally by k -algebra automorphisms. (So G acts on $\mathrm{Spec}(A)$.) Then the cohomology ring $H^*(G, A)$ is finitely generated as a k -algebra. Here, as in [9, I.4], we use the cohomology introduced by Hochschild, also known as ‘rational cohomology’.

In [13] we have shown that SL_2 over a field of positive characteristic has property (CFG), and in [14] we proved that SL_3 over a field of characteristic two has property (CFG). We conjecture that every reductive linear algebraic group has property (CFG). In this paper we show that this is at least a good heuristic principle: We derive one of the consequences of (CFG) for any simply connected semisimple linear algebraic group G that satisfies the following

Hypothesis 1.1 *Assume that for every fundamental weight ϖ_i the symmetric algebra $S^*(\nabla_G(\varpi_i))$ on the fundamental representation $\nabla_G(\varpi_i)$ has a good filtration.*

Recall that this hypothesis is satisfied if $p \geq \max_i(\dim(\nabla_G(\varpi_i)))$, by [1, 4.1(5) and 4.3(1)]. This inequality is not necessary. For instance, SL_n satisfies the hypothesis for $n \leq 5$, by [13, Lemma 3.2]. When $p = 2$, the hypothesis does not hold for SL_n with $n \geq 6$, by [13, 3.3].

In the sequel let G be a connected reductive linear algebraic group over a field k of characteristic $p > 0$ with simply connected commutator subgroup for which hypothesis 1.1 holds. Let A be a finitely generated commutative k -algebra on which G acts rationally by k -algebra automorphisms. Let M be a noetherian A -module on which G acts compatibly. This means that the structure map $A \otimes M \rightarrow M$ is a G -module map. Our main result is

Theorem 1.2 *If A has good filtration, then M has finite good filtration dimension and each $H^i(G, M)$ is a noetherian A^G -module.*

When $A = k$ the theorem goes back to [5] and does not need hypothesis 1.1. Unlike the proofs in [13] and [14], the proof of our theorem does not involve any cohomology of finite group schemes and is thus independent of the work of Friedlander and Suslin [6]. But without their work we would not have guessed the theorem. For clarity we will pull some material of [13] free from finite group schemes.

2 Recollections

Some unexplained notations, terminology, properties, ... can be found in [9]. We choose a Borel group $B^+ = TU^+$ and the opposite Borel group B^- . The roots of B^+ are positive. If $\lambda \in X(T)$ is dominant, then $\text{ind}_{B^-}^G(\lambda)$ is the ‘dual Weyl module’ or ‘costandard module’ $\nabla_G(\lambda)$ with highest weight λ . The formula $\nabla_G(\lambda) = \text{ind}_{B^-}^G(\lambda)$ just means that $\nabla_G(\lambda)$ is obtained from the Borel-Weil construction: $\nabla_G(\lambda)$ equals $H^0(G/B^-, \mathcal{L})$ for a certain line bundle on the flag variety G/B^- . In a *good filtration* $0 = V_{-1} \subseteq V_0 \subseteq V_1 \dots$ of a G -module $V = \bigcup_i V_i$ the nonzero layers V_i/V_{i-1} are of the form $\nabla_G(\mu)$. As in [12] we will actually also allow a layer to be a direct sum of any number of copies of the same $\nabla_G(\mu)$, cf. [9, II.4.16 Remark 1]. This is much more convenient when working with infinite dimensional G -modules. It is shown

in [4] that a module of countable dimension that has a good filtration in our sense also has a filtration that is a good filtration in the old sense. Note that the module M in our theorem has countable dimension. It would do little harm to restrict to modules of countable dimension throughout.

If V is a G -module, and $m \geq -1$ is an integer so that $H^{m+1}(G, \nabla_G(\mu) \otimes V) = 0$ for all dominant μ , then we say as in [5] that V has *good filtration dimension* at most m . The case $m = 0$ corresponds with V having a good filtration. And for $m \geq 0$ it means that V has a resolution

$$0 \rightarrow V \rightarrow N_0 \rightarrow \cdots \rightarrow N_m \rightarrow 0$$

in which the N_i have good filtration, in our sense. We say that V has good filtration dimension precisely m , notation $\dim_{\nabla}(V) = m$, if m is minimal so that V has good filtration dimension at most m . In that case $H^{i+1}(G, \nabla_G(\mu) \otimes V) = 0$ for all dominant μ and all $i \geq m$. In particular $H^{i+1}(G, V) = 0$ for $i \geq m$. If there is no finite m so that $\dim_{\nabla}(V) = m$, then we put $\dim_{\nabla}(V) = \infty$.

2.1 Filtrations

For simplicity assume also that G is semisimple. (Until remark 3.1.) If V is a G -module, and λ is a dominant weight, then $V_{\leq \lambda}$ denotes the largest G -submodule all whose weights μ satisfy $\mu \leq \lambda$ in the dominance partial order [9, II.1.5]. For instance, $V_{\leq 0}$ is the module of invariants V^G . Similarly $V_{< \lambda}$ denotes the largest G -submodule all whose weights μ satisfy $\mu < \lambda$. As in [12], we form the $X(T)$ -graded module

$$\mathrm{gr}_{X(T)} V = \bigoplus_{\lambda \in X(T)} V_{\leq \lambda} / V_{< \lambda}.$$

Each $V_{\leq \lambda} / V_{< \lambda}$, or $V_{\leq \lambda / < \lambda}$ for short, has a B^+ -socle $(V_{\leq \lambda / < \lambda})^U = V_{\lambda}^U$ of weight λ . We always view V^U as a B^- -module through restriction (inflation) along the homomorphism $B^- \rightarrow T$. Then $V_{\leq \lambda / < \lambda}$ embeds naturally in its ‘good filtration hull’ $\mathrm{hull}_{\nabla}(V_{\leq \lambda / < \lambda}) = \mathrm{ind}_{B^-}^G V_{\lambda}^U$. This good filtration hull has the same B^+ -socle and by Polo it is the injective hull in the category \mathcal{C}_{λ} of G -modules N that satisfy $N = N_{\leq \lambda}$. Compare [12, 3.1.10].

We convert the $X(T)$ -graded module $\mathrm{gr}_{X(T)} V$ to a \mathbb{Z} -graded module through an additive height function $\mathrm{ht} : X(T) \rightarrow \mathbb{Z}$, defined by $\mathrm{ht} = 2 \sum_{\alpha > 0} \alpha^{\vee}$, the sum being over the positive roots. (Our ht is twice the one

used by Grosshans [7], because we prefer to get even degrees rather than just integer degrees.) The Grosshans graded module is now

$$\mathrm{gr} V = \bigoplus_{i \geq 0} \mathrm{gr}_i V,$$

with

$$\mathrm{gr}_i V = \bigoplus_{\mathrm{ht}(\lambda)=i} V_{\leq \lambda / < \lambda}.$$

In other words, if one puts

$$V_{\leq i} := \sum_{\mathrm{ht}(\lambda) \leq i} V_{\leq \lambda},$$

then $\mathrm{gr} V$ is the associated graded of the filtration $V_{\leq 0} \subseteq V_{\leq 1} \cdots$.

Let us apply the above to our finitely generated commutative k -algebra with G -action A . The Grosshans graded algebra $\mathrm{gr} A$ embeds in a good filtration hull, which Grosshans calls R , and which we call $\mathrm{hull}_{\nabla}(\mathrm{gr} A)$,

$$\mathrm{hull}_{\nabla}(\mathrm{gr} A) := \mathrm{ind}_{B^-}^G A^U = \bigoplus_i \bigoplus_{\mathrm{ht}(\lambda)=i} \mathrm{hull}_{\nabla}(A_{\leq \lambda} / A_{< \lambda}).$$

Grosshans [7] shows that A^U , $\mathrm{gr} A$, $\mathrm{hull}_{\nabla}(\mathrm{gr} A)$ are finitely generated k -algebras with $\mathrm{hull}_{\nabla}(\mathrm{gr} A)$ finite over $\mathrm{gr} A$. Mathieu studied $\mathrm{gr} A$ and $\mathrm{hull}_{\nabla}(\mathrm{gr} A)$ earlier in [11].

Example 2.2 Consider the multicone [10]

$$k[G/U] := \mathrm{ind}_U^G k = \mathrm{ind}_{B^+}^G \mathrm{ind}_U^{B^+} k = \mathrm{ind}_{B^+}^G k[T] = \bigoplus_{\lambda \text{ dominant}} \nabla_G(\lambda).$$

It is its own Grosshans graded ring. Recall [10] that it is generated as a k -algebra by the finite dimensional sum of the $\nabla_G(\varpi_i)$, where ϖ_i denotes the i th fundamental weight.

Lemma 2.3 *Let A have a good filtration, so that $\mathrm{gr} A = \mathrm{hull}_{\nabla}(\mathrm{gr} A)$. Let $R = \bigoplus_i R_i$ be a graded algebra with G -action such that $R_i = (R_i)_{\leq i}$. Then every T -equivariant graded algebra homomorphism $R^U \rightarrow (\mathrm{gr} A)^U$ extends uniquely to a G -equivariant graded algebra homomorphism $R \rightarrow (\mathrm{gr} A)$.*

Proof Use that $\text{hull}_{\nabla}(\text{gr } A)$ is an induced module. \square

2.4 A graded polynomial $G \times D$ -algebra with good filtration

We now extract a construction from [13]. It is hidden in the study of a Hochschild-Serre spectral sequence which in the present situation would correspond with the case where as normal subgroup one takes the trivial subgroup!

As the algebra $(\text{gr } A)^U$ is finitely generated, it is also generated by finitely many weight vectors. Consider one such weight vector v , say of weight λ . Clearly λ is dominant. If $\lambda = 0$, map a polynomial ring $P_v := k[x]$ with trivial G -action to $\text{gr } A$ by substituting v for x . Also put $D_v := 1$. Next assume $\lambda \neq 0$. Let ℓ be the rank of G . Define a T -action on the $X(T)$ -graded algebra

$$P = \bigotimes_{i=1}^{\ell} S^*(\nabla_G(\varpi_i))$$

by letting T act on $\bigotimes_{i=1}^{\ell} S^{m_i}(\nabla_G(\varpi_i))$ through weight $\sum_i m_i \varpi_i$. So now we have a $G \times T$ -action on P . Observe that by our key hypothesis 1.1 and the tensor product property [9, Ch. G] the polynomial algebra P has a good filtration for the G -action. Let D be the scheme theoretic kernel of λ . So D has character group $X(D) = X(T)/\mathbb{Z}\lambda$ and $D = \text{Diag}(X(T)/\mathbb{Z}\lambda)$ in the notations of [9, I.2.5]. The subalgebra $P^{1 \times D}$ is a graded algebra with good filtration such that its subalgebra $P^{U \times D}$ contains a polynomial algebra on one generator x of weight $\lambda \times \lambda$. In fact, this polynomial subalgebra contains all the weight vectors in $P^{U \times D}$ of weight $\mu \times \nu$ with $\text{ht}(\mu) \geq \text{ht}(\nu)$. The other weight vectors in $P^{U \times D}$ also have weight of the form $\mu \times \nu$ with ν a multiple of λ . These other weight vectors span an ideal in $P^{U \times D}$. Now assume A has a good filtration. By lemma 2.3 one easily constructs a G -equivariant algebra homomorphism $P^{1 \times D} \rightarrow \text{gr } A$ that maps x to v . Write it as $P_v^{1 \times D_v} \rightarrow \text{gr } A$, to stress the dependence on v .

As new P we take the tensor product of the finitely many P_v and as diagonalized group D we take the direct product of the D_v . Then we have a graded algebra map $P^D \rightarrow \text{gr } A$. It is surjective because its image has good filtration ([9, Ch. A]) and contains $(\text{gr } A)^U$. The $G \times D$ -algebra P is an example of what we called in [13] a graded polynomial $G \times D$ -algebra with good filtration. We have proved

Lemma 2.5 *If A has a good filtration, then there is a graded polynomial $G \times D$ -algebra P with good filtration and a graded G -equivariant surjection $P^D \rightarrow \text{gr } A$.*

Now recall M is a noetherian A -module on which G acts compatibly, meaning that the structure map $A \otimes M \rightarrow M$ is a map of G -modules. Form the ‘semi-direct product ring’ $A \ltimes M$ whose underlying G -module is $A \oplus M$, with product given by $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1)$. By Grosshans $\text{gr}(A \ltimes M)$ is a finitely generated algebra, so we get

Lemma 2.6 *$\text{gr } M$ is a noetherian $\text{gr } A$ -module.*

This is of course very reminiscent of the proof of the lemma [8, Thm. 16.9] telling that M^G is a noetherian module over the finitely generated k -algebra A^G . We will tacitly use its counterpart for diagonalized actions, cf. [2], [9, I.2.11].

Taking things together we learn that if A has a good filtration, then $P \otimes_{P^D} \text{gr } M$ is what we called in [13] a finite graded P -module. Lemma [13, Lemma 3.7] then tells us

Lemma 2.7 *Let A have good filtration. Then $P \otimes_{P^D} \text{gr } M$ has finite good filtration dimension and each $H^i(G, P \otimes_{P^D} \text{gr } M)$ is a noetherian P^G -module.*

Extend the D -action on P to $P \otimes_{P^D} \text{gr } M$ by using the trivial action on the second factor. Then we have a $G \times D$ -module structure on $P \otimes_{P^D} \text{gr } M$. As D is diagonalized, P^D is a direct summand of P as a P^D -module [9, I.2.11] and $(P \otimes_{P^D} \text{gr } M)^{1 \times D} = \text{gr } M$ is a direct summand of the G -module $P \otimes_{P^D} \text{gr } M$. It follows that $\text{gr } M$ also has finite good filtration dimension and it follows that each $H^i(G, P \otimes_{P^D} \text{gr } M)^{1 \times D} = H^i(G, \text{gr } M)$ is a noetherian $P^{G \times D}$ -module. But the action of $P^{G \times D}$ on $\text{gr } M$ factors through $(\text{gr } A)^G$, so we see that each $H^i(G, \text{gr } M)$ is a noetherian $(\text{gr } A)^G$ -module. And one always has $(\text{gr } A)^G = (\text{gr}_0 A)^G = A^G$. We conclude

Lemma 2.8 *Let A have good filtration. Then $\text{gr } M$ has finite good filtration dimension and each $H^i(G, \text{gr } M)$ is a noetherian A^G -module.*

3 Degradation

We still have to get rid of the grading. The filtration $M_{\leq 0} \subseteq M_{\leq 1} \cdots$ induces a filtration of the Hochschild complex [9, I.4.14] whence a spectral sequence

$$E(M) : E_1^{ij} = H^{i+j}(G, \text{gr}_{-i} M) \Rightarrow H^{i+j}(G, M).$$

It lives in an unusual quadrant.

Assume that A has good filtration. Then by Lemma 2.8 $E_1(M)$ is a finitely generated A^G -module. So the spectral sequence lives in only finitely many bidegrees (i, j) . Thus there is the same kind of convergence as one would have in a more common quadrant.

Choose A^G as ring of operators to act on the spectral sequence $E(M)$. As $E_1(M)$ is a noetherian A^G -module, it easily follows (even without the spectral sequence) that $H^*(G, M)$ is a noetherian A^G -module. To finish the proof of the theorem, we note that $A \otimes k[G/U]$ is also a finitely generated algebra with a good filtration and that $M \otimes k[G/U]$ is a noetherian module over it. So what we have just seen tells that $H^*(G, M \otimes k[G/U])$ is a noetherian $(A \otimes k[G/U])^G$ -module. In particular, there is an $m \geq -1$ so that $H^{m+1}(G, M \otimes k[G/U]) = 0$. \square

Remark 3.1 Somewhere along the way we made the simplifying assumption that G is semisimple. So for the original G we have now proved that M has finite good filtration dimension with respect to the commutator subgroup H of G . But that is the same as having finite good filtration dimension with respect to G . Also, the fact that $H^i(H, M)$ is a noetherian A^H -module implies that $H^i(G, M)$ is a noetherian A^G -module by taking invariants under the diagonalizable center $Z(G)$.

Remark 3.2 We did not prove that M has a finite resolution by noetherian A -modules with compatible G -action and good filtration. We do not know how to start. One may embed M into the A -module $M \otimes k[G]$ with compatible G -action. It has good filtration, but it is not noetherian as an A -module.

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